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NORMAL TERNARY CONTINUED FRACTION EXPANSIONS FOR THE CUBE ROOTS OF INTEGERS.*

BY P. H. DAUS.

INTRODUCTION.

The problem of finding rational approximations to a cubic irrationality by means of a periodic expansion was first attempted and partially solved by Jacobi† in 1868. Jacobi devised an extension of continued fractions, which enabled him to determine rational approximations to the mutual ratios of three numbers of such a nature that any approximation (A_n, B_n, C_n) could be expressed in terms of the three preceding ones by means of the recursion formulæ,

$$(A)\qquad \begin{aligned} A_n &= p_n A_{n-1} + q_n A_{n-2} + A_{n-3}, \\ B_n &= p_n B_{n-1} + q_n B_{n-2} + B_{n-3}, \\ C_n &= p_n C_{n-1} + q_n C_{n-2} + C_{n-3}. \end{aligned}$$

He further showed that if the three numbers be taken $1 : \theta : a + b\theta + c\theta^2$, where θ is the real root of a rational cubic equation of negative discriminant, that is, one with one real root, the expansion may become periodic, and in actual numerical cases does so. It has been proved by Bachman,‡ however, that this periodicity, using Jacobi's method of selecting (p_n, q_n), can not exist unless a certain limiting condition be satisfied. Berwick§ has obtained periodic expansions for cubic irrationalities, but his algorithm differs from Jacobi's, and has the disadvantage that the transformations involved are not necessarily unimodular and may be singular. Lehmer,¶ instead of starting with a cubic irrationality, started with a periodic expansion and found associated with it a definite cubic irrationality.

Closely allied to this problem, in the case when $\theta = \sqrt[3]{D}$, is the solution of the Pellian cubic

$$(B)\qquad F(x, y, z) = x^3 + Dy^3 + D^2z^3 - 3Dxyz = m.$$

The value of $F(x, y, z)$ can be expressed in several other ways which we shall find useful. As a determinant

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* Since this article was submitted for publication, the results have been extended to the roots of the more general cubic equation $x^3 + px^2 + qx + r = 0$. These results will be the subject of a later article.

† Jacobi, C. G. J., *Ges. Werke*, VI (385–426).

‡ Bachman, P., *Crelle*, Vol. 75 (1873) (25–34).

§ Berwick, W. E. H., *Proc. London Math. Soc.* (Ser. 2), Vol. 12 (1912) (393–429).

¶ Lehmer, D. N., *Proc. Nat. Acad. of Sciences*, Vol. 4, Dec., 1918 (360–364).

$$(C) \qquad F(x, y, z) = \begin{vmatrix} x & Dz & Dy \\ y & x & Dz \\ z & y & x \end{vmatrix}.$$

Or as a product of three conjugate factors

$$(D) \quad F(x, y, z) = (x + y\theta + z\theta^2)(x + \omega\theta y + \omega^2\theta^2 z)(x + \omega^2\theta y + \omega\theta^2 z),$$

where ω is an imaginary cube root of unity. Dickson* in his History of the Theory of Numbers describes briefly the work done along this line. Tables of solutions have been computed by Meissel† for $D < 82$ (although he did not publish his table) and an incomplete table by Wolfe‡ for $D < 100$. This latter table has been used in computations made for this paper.

The object of this paper is to show that by a suitable choice of (p_n, q_n), Jacobi's method may be modified, and that the modified expansions bear close analogies to ordinary continued fractions. This leads to a definition of normal ternary continued fractions and an attempt to compute a table of such expansions for the cube roots of integers.

If u_1, v_1, w_1 be any three numbers, we define a ternary continued fraction expansion for them by the equations

$$u_{n+1} = v_n - p_n u_n, \qquad v_{n+1} = w_n - q_n u_n, \qquad w_{n+1} = u_n,$$

where (u_n, v_n, w_n) is called the nth complete quotient set, (p_n, q_n) the nth partial quotient set and (A_n, B_n, C_n), as defined by equations (A) of the introduction, the nth convergent set to the ternary continued fraction

$$\left(\frac{v_1}{u_1}, \frac{w_1}{u_1}\right) = (p_1, q_1;\ p_2, q_2;\ p_3, q_3;\ \cdots).$$

The following set of relations connecting the first and general complete quotient sets is fundamental.

THEOREM I. *If u_1, v_1, w_1, any three numbers, be expanded into a ternary continued fraction, and if $\sigma_{1,\,n} = v_n/u_n$, $\sigma_{2,\,n} = w_n/u_n$, and (A_n, B_n, C_n) be the nth convergent set, then*

$$\sigma_{1,\,1} = \frac{B_n\sigma_{2,\,n+1} + B_{n-1}\sigma_{1,\,n+1} + B_{n-2}}{A_n\sigma_{2,\,n+1} + A_{n-1}\sigma_{1,\,n+1} + A_{n-2}}$$

and

$$\sigma_{2,\,1} = \frac{C_n\sigma_{2,\,n+1} + C_{n-1}\sigma_{1,\,n+1} + C_{n-2}}{A_n\sigma_{2,\,n+1} + A_{n-1}\sigma_{1,\,n+1} + A_{n-2}}.$$

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* Dickson, L. E., Vol. II (593–595).

† Meissel, E., Program, Kiel, 1891.

‡ Wolfe, C. (work done at the Univ. of Calif.; not yet published).

The defining equations may be written by combining them,

$$
\begin{array}{rrcl}
u_n = & & & w_{n+1} \\
v_n = & u_{n+1} & & + p_n w_{n+1} \\
w_n = & & v_{n+1} & + q_n w_{n+1}
\end{array} \tag{1}
$$

defining a unimodular collineation, and we can write

$$
\begin{aligned}
u_1 &= A_{n-2}u_{n+1} + A_{n-1}v_{n+1} + A_n w_{n+1} \\
v_1 &= B_{n-2}u_{n+1} + B_{n-1}v_{n+1} + B_n w_{n+1} \\
w_1 &= C_{n-2}u_{n+1} + C_{n-1}v_{n+1} + C_n w_{n+1}
\end{aligned} \tag{2}
$$

where

$$
\begin{vmatrix} A_{n-2} & A_{n-1} & A_n \\ B_{n-2} & B_{n-1} & B_n \\ C_{n-2} & C_{n-1} & C_n \end{vmatrix} = \prod_1^n \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & p_n \\ 0 & 1 & q_n \end{vmatrix} .
$$

Now

$$
\begin{vmatrix} A_{n-2} & A_{n-1} & A_n \\ B_{n-2} & B_{n-1} & B_n \\ C_{n-2} & C_{n-1} & C_n \end{vmatrix} \times \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & p_{n+1} \\ 0 & 1 & q_{n+1} \end{vmatrix} = \begin{vmatrix} A_{n-1} & A_n & A_{n-2}+p_{n+1}A_{n-1}+q_{n+1}A_n \\ B_{n-1} & B_n & B_{n-2}+p_{n+1}B_{n-1}+q_{n+1}B_n \\ C_{n-1} & C_n & C_{n-2}+p_{n+1}C_{n-1}+q_{n+1}C_n \end{vmatrix} ,
$$

from which we have the recursion formulæ for convergent sets

$$
\begin{aligned}
A_n &= p_n A_{n-1} + q_n A_{n-2} + A_{n-3} \\
B_n &= p_n B_{n-1} + q_n B_{n-2} + B_{n-3} \\
C_n &= p_n C_{n-1} + q_n C_{n-2} + C_{n-3}
\end{aligned} \tag{3}
$$

with the initial conditions

$$
\begin{vmatrix} A_{-2} & A_{-1} & A_0 \\ B_{-2} & B_{-1} & B_0 \\ C_{-2} & C_{-1} & C_0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} .
$$

In actual calculations we interchange rows and columns and compute successive convergents by the recursion formulæ (3). From equations (2) by division to make the expressions non-homogeneous, we have the equations stated in the theorem.

By solving equations (2) for u_{n+1}, v_{n+1}, and w_{n+1} we have

$$
\begin{aligned}
u_{n+1} &= c_{n-2}w_1 + b_{n-2}v_1 + a_{n-2}u_1, \\
v_{n+1} &= c_{n-1}w_1 + b_{n-1}v_1 + a_{n-1}u_1, \\
w_{n+1} &= c_n\ \ w_1 + b_n\ \ v_1 + a_n\ \ u_1,
\end{aligned} \tag{4}
$$

where a_n, b_{n-1}, etc., are the cofactors of the corresponding elements of the determinant in (2). Dividing v_{n+1} and w_{n+1} respectively by u_{n+1}, we obtain as a

COROLLARY. *Under the conditions of theorem I*

$$
\begin{aligned}
\sigma_{2,\,n+1} &= \frac{c_n\sigma_{2,\,1} + b_n\sigma_{1,\,1} + a_n}{c_{n-2}\sigma_{2,\,1} + b_{n-2}\sigma_{1,\,1} + a_{n-2}} ; \\
\sigma_{1,\,n+1} &= \frac{c_{n-1}\sigma_{2,\,1} + b_{n-1}\sigma_{1,\,1} + a_{n-1}}{c_{n-2}\sigma_{2,\,1} + b_{n-2}\sigma_{1,\,1} + a_{n-2}} .
\end{aligned} \tag{5}
$$

The equations deduced in theorem I are independent of the choice of (p_n, q_n). In order to extend the continued fraction algorithm it is desirable to have $A_n < B_n < C_n$, and to have the A's, B's and C's each form a positive increasing series of integers. Such an expansion will be called *regular*. The defining equations may be written in the form

$$\sigma_{1,\,n} = p_n + \frac{1}{\sigma_{2,\,n+1}}\,; \qquad \sigma_{2,\,n} = q_n + \frac{\sigma_{1,\,n+1}}{\sigma_{2,\,n+1}}\,.$$

Jacobi* chose p_n and q_n as $[\sigma_{1,\,n}]$ and $[\sigma_{2,\,n}]$ respectively, where the symbol $[x]$ denotes the greatest integer in x. We notice that the equation defining $\sigma_{1,\,n}$ is the defining equation for ordinary continued fractions, while that for $\sigma_{2,\,n}$ is not. We note also that the expansion may remain regular under other conditions, which we state as

THEOREM II. *If u_1, v_1, w_1, three positive numbers, such that $u_1 < v_1 < w_1$, be expanded into a ternary continued fraction, the laws of selecting (p_n, q_n) being $p_n = [v_n/u_n]$, $0 < q_n \leqq w_n/u_n$, and $[q_1 \geqq p_1]$, then the expansion is regular as defined above.*

This selection of (p_n, q_n) is such that u_n, v_n, and w_n are always positive, as well as p_n and q_n, and from the recursion formulæ (3) it follows that A_n, B_n, and C_n each form a positive and increasing series of numbers. Since $A_1 = 1$, $B_1 = p_1$, and $C_1 = q_1$, we have

$$A_1 \leqq B_1 \leqq C_1.$$

Also $A_2 = q_2$, $B_2 = p_1q_2 + 1$, $C_2 = q_1q_2 + p_2$, and since $p_1 \geqq 1$, $p_2 \geqq 0$, and $q_1 \geqq p_1$, we have

$$A_2 < B_2 \leqq C_2,$$

the equality sign holding if $p_2 = 0$, $q_2 = 1$ and $q_1 = p_1 + 1$, for if $q_1 > p_1$, then $q_1 \geqq p_1 + 1$ and $C_2 \geqq (p_1 + 1)q_2 + p_2 \geqq B_2$. The equality sign also holds if $q_1 = p_1$, for in that case (since $w_1 > u_1$) $v_2 > u_2$ and consequently $p_2 \geqq 1$ and $C_2 \geqq q_1q_2 + 1 \geqq B_2$. Now

$$\begin{aligned} A_3 &= q_3A_2 + p_3A_1 + 0,\\ B_3 &= q_3B_2 + p_3B_1 + 0,\\ C_3 &= q_3C_2 + p_3C_1 + 1. \end{aligned}$$

Therefore $A_3 < B_3 < C_3$, and in general $A_n < B_n < C_n$ for $n \geqq 3$.

Although the above conditions are sufficient, they are not necessary. One exception is that q_n may be zero, provided that p_n be large enough.

An important case of a regular expansion is one which may be considered as an extension of an ordinary continued fraction, in which the partial quotients are palindromic.* In this case, however, we define a skew-palindromic expansion as one in which the q's are palindromic and the p's, omitting the first, are also palindromic.

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* Jacobi, C. G. J., *Ges. Werke*, VI (385–426).

THEOREM III. *If* u_1, v_1, w_1, *three rational numbers, relatively prime, be expanded into a regular ternary continued fraction, then the necessary and sufficient condition that the series* $u_1, u_2, \cdots u_n$ *be the series* $A_n, A_{n-1}, \cdots A_1$ *(i.e.,* $A_k = u_{n-k+1}$*), where* (A_n, B_n, C_n) *is the last convergent set, is that after the first partial quotient set, the expansion be skew-palindromic.*

By means of equations (1) we have

$$u_k = w_{k+1} = q_{k+1}u_{k+1} + v_{k+2}$$

and consequently

$$u_k = q_{k+1}u_{k+1} + p_{k+2}u_{k+2} + u_{k+3}. \tag{6}$$

The conditions imposed on p_k and q_k may be written

$$p_k = p_{n-k+3}; \qquad q_k = q_{n-k+2}. \tag{7}$$

Since we have selected u_1, v_1, and w_1 rational and relatively prime, we have $u_n = 1$, $v_n = p_n$, and $w_n = q_n$.

In general

$$A_{n-k+1} = q_{n-k+1}A_{n-k} + p_{n-k+1}A_{n-k-1} + A_{n-k-2}, \tag{8}$$

and in particular $A_0 = 0$; $A_1 = 1 = u_n$; $A_2 = q_2 = w_n = u_{n-1}$. The proof is by induction. Assume $A_k = u_{n-k+1}$ for all values up to and including k. Then

$$A_{k+1} = q_{k+1}A_k + p_{k+1}A_{k-1} + A_{k-2} \tag{8'}$$

and

$$u_{n-k} = q_{n-k+1}u_{n-k+1} + p_{n-k+2}u_{n-k+2} + u_{n-k+3}. \tag{6'}$$

And by comparison, using our assumption and equations (7), we get

$$A_{k+1} = u_{n-(k+1)+1} = u_{n-k}.$$

And since $A_k = u_{n-k+1}$ for $k = 0, 1, 2$, it follows that it is true for all values of k.

Conversely if $A_k = u_{n-k+1}$ for all values of k, then equations (7) are true. For by subtracting equation (6′) from (8′) we have, using this assumed equation,

$$0 = A_k(q_{k+1} - q_{n-k+1}) + (p_{k+1} - p_{n-k+2})A_{k-1}. \tag{9}$$

Since $A_2 = q_2$, $q_n = u_{n-1}$ and we have assumed $A_2 = u_{n-1}$, it follows from (9), by putting $k = n - 1$, that $p_3 = p_n$. Assume $q_k = q_{n-k+2}$ and $p_{k+1} = p_{n-k+2}$ for all values up to k. It follows by comparing equations (6′) and

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* A sequence of numbers is said to be palindromic, when it reads the same backwards as forwards.

(8′) that

$$q_{k+1} = q_{n-(k+1)+2}$$

and from (9)

$$p_{(k+1)+1} = p_{n-(k+1)+2}.$$

Hence the theorem is true.

As a consequence of this theorem, we state as a

COROLLARY. *Under the hypothesis of theorem III*

$$B_n = p_1 A_n + A_{n-1} \qquad \textit{and} \qquad C_n = q_1 A_n + p_2 A_{n-1} + A_{n-2}.$$

By using the particular values $w_{n+1} = 1$, $v_{n+1} = 0$ and $u_{n+1} = 0$ in equations (2), we obtain

$$A_n = u_1, \qquad B_n = v_1, \qquad C_n = w_1.$$

And since $v_1 = p_1 u_1 + u_2$ and $w_1 = q_1 u_1 + p_2 u_2 + u_3$, we obtain the equations stated in the corollary.

After these preliminary considerations we come to the following theorem, which is fundamental in the considerations of periodic expansions.

THEOREM IV. *If in a ternary continued fraction, the expansion become periodic after a finite number of terms, then $\sigma_{1,\,1}$ and $\sigma_{2,\,1}$ are roots of cubic equations with rational coefficients.*

If we think of the σ's as recurring, obviously the p's and q's recur, and conversely, if we think of the p's and q's as recurring, then the σ's recur also, for it does not matter which period we take to determine these cubic equations.

Consider first the case where the expansion is purely periodic, i.e., where $\sigma_{1,\,1} = \sigma_{1,\,n+1}$ and $\sigma_{2,\,1} = \sigma_{2,\,n+1}$. Then from theorem I we have

$$\sigma_{1,\,1} = \frac{B_n\sigma_{2,\,1} + B_{n-1}\sigma_{1,\,1} + B_{n-2}}{A_n\sigma_{2,\,1} + A_{n-1}\sigma_{1,\,1} + A_{n-2}}; \qquad \sigma_{2,\,1} = \frac{C_n\sigma_{2,\,1} + C_{n-1}\sigma_{1,\,1} + C_{n-2}}{A_n\sigma_{2,\,1} + A_{n-1}\sigma_{1,\,1} + A_{n-2}}.$$

Solving the first of these equations for $\sigma_{2,\,1}$ and substituting in the second, we get

$$\sigma_{2,\,1} = \frac{-A_{n-1}\sigma_{1,\,1}^2 + (B_{n-1} - A_{n-2})\sigma_{1,\,1} + B_{n-2}}{A_n\sigma_{1,\,1} - B_n}$$

$$= \frac{C_n\left[\dfrac{-A_{n-1}\sigma_{1,\,1}^2 + (B_{n-1} - A_{n-2})\sigma_{1,\,1} + B_{n-2}}{A_n\sigma_{1,\,1} - B_n}\right] + C_{n-1}\sigma_{1,\,1} + C_{n-2}}{A_n\left[\dfrac{-A_{n-1}\sigma_{1,\,1}^2 + (B_{n-1} - A_{n-2})\sigma_{1,\,1} + B_{n-2}}{A_n\sigma_{1,\,1} - B_n}\right] + A_{n-1}\sigma_{1,\,1} + A_{n-2}}.$$

And by multiplying up and combining terms we write

$$\sigma_{2,\,1} = \frac{b_{n-2}\sigma_{1,\,1}^2 + (a_{n-2} - b_{n-1})\sigma_{1,\,1} - a_{n-1}}{-c_{n-2}\sigma_{1,\,1} + c_{n-1}},$$

and equating values of $\sigma_{2,\,1}$

$$[-A_{n-1}\sigma_{1,\,1}^2 + (B_{n-1} - A_{n-2})\sigma_{1,\,1} + B_{n-2}][-c_{n-2}\sigma_{1,\,1} + c_{n-1}]$$
$$= [A_n\sigma_{1,\,1} - B_n][b_{n-2}\sigma_{1,\,1}^2 + (a_{n-2} - b_{n-1})\sigma_{1,\,1} - a_{n-1}].$$

Finally

$$E_1\sigma_{1,\,1}^3 + F_1\sigma_{1,\,1}^2 + G_1\sigma_{1,\,1} + H_1 = 0, \tag{10}$$

where $E_1 = A_nb_{n-2} - A_{n-1}c_{n-2}$.

$$F_1 = A_{n-1}c_{n-1} + B_{n-1}c_{n-2} - A_{n-2}c_{n-2} - B_nb_{n-2} + A_na_{n-2} - A_nb_{n-1}.$$
$$G_1 = B_{n-2}c_{n-2} - B_{n-1}c_{n-1} + A_{n-2}c_{n-1} - B_na_{n-2} + B_nb_{n-1} - A_na_{n-1}.$$
$$H_1 = B_na_{n-1} - B_{n-2}c_{n-1}.$$

In a similar fashion by eliminating $\sigma_{1,\,1}$, we get the following equation:

$$E_2\sigma_{2,\,1}^3 + F_2\sigma_{2,\,1}^2 + G_2\sigma_{2,\,1} + H_2 = 0, \tag{11}$$

where $E_2 = -E_1$.

$$F_2 = A_nb_n + C_nb_{n-2} - A_{n-2}b_{n-2} - C_{n-1}c_{n-2} + A_{n-1}a_{n-2} - A_{n-1}c_n.$$
$$G_2 = C_{n-2}b_{n-2} - C_nb_n + A_{n-2}b_n - C_{n-1}a_{n-2} + C_{n-1}c_n - A_{n-1}a_n.$$
$$H_2 = C_{n-1}a_n - C_{n-2}b_n.$$

These equations have previously been obtained by Lehmer,* by another method.

If the expansion is not purely periodic, but say the κth set of σ's is the first to recur, then $\sigma_{1,\,\kappa}$ and $\sigma_{2,\,\kappa}$ are roots of cubic equations, and since $\sigma_{1,\,1}$ and $\sigma_{2,\,1}$ are linear fractional functions of $\sigma_{1,\,\kappa}$, $\sigma_{2,\,\kappa}$, $\sigma_{1,\,1}$ and $\sigma_{2,\,1}$ are also roots of cubic equations.

In what follows we shall confine our attention to the case where $\sigma_{1,\,1} = \theta$ and $\sigma_{2,\,1} = \theta^2$, θ being the real cube root of D, an integer, but not a perfect cube. It will be convenient to use as well as the complete quotient set (u_n, v_n, w_n) composed of linear functions of θ and θ^2, the rationalized complete quotient set

$$(\bar{u}_n, \bar{v}_n, \bar{w}_n) = (\alpha_n, \alpha_n' + \beta_n'\theta + \gamma_n'\theta^2, \alpha_n'' + \beta_n''\theta + \gamma_n''\theta^2),$$

where $\alpha_n =$ Norm of u_n and

$$(\alpha_n' + \beta_n'\theta + \gamma_n'\theta^2) = \frac{v_nN(u_n)}{u_n},$$

and

$$(\alpha_n'' + \beta_n''\theta + \gamma_n''\theta^2) = \frac{w_nN(u_n)}{u_n}.$$

The quantities $\bar{u}_n$, $\bar{v}_n$, and $\bar{w}_n$ may have a common integral factor, which is not to be removed as was done by Jacobi.† (See the illustrative example after theorem IX.)

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* Lehmer, D. N., *Proc. Nat. Acad. Sciences*, Vol. 4, Dec., 1918 (360–364).

† Jacobi, C. G. J., *Ges. Werke*, VI (385–426).

THEOREM V. *If* 1, θ, θ^2 *expand into a ternary continued fraction which ultimately becomes periodic, and if* m_k *be the value of the Pellian cubic for* (A_k, B_k, C_k), *then the series* m_k *also becomes periodic.*

Suppose that there are t non-recurring partial and complete quotient sets, after which the expansion becomes periodic, the period containing n terms. Then $u_{t+k} : v_{t+k} : w_{t+k} = u_{t+k+n} : v_{t+k+n} : w_{t+k+n}$ (for all positive k's, since the ratios involved are the roots obtained from cubics computed from identical periods). Since

$$w_{t+k} = u_{t+k-1} \qquad \text{and} \qquad w_{t+k+n} = u_{t+k+n-1}$$

we can write

$$\frac{u_{t+n+1}}{u_{t+n}} = \frac{u_{t+1}}{u_t}, \qquad \frac{u_{t+n+1}}{u_{t+1}} = \frac{v_{t+n+1}}{v_{t+1}},$$

$$\frac{u_{t+n+2}}{u_{t+n+1}} = \frac{u_{t+2}}{u_{t+1}}, \qquad \frac{u_{t+n+2}}{u_{t+2}} = \frac{v_{t+n+2}}{v_{t+2}},$$

etc.

It follows that

$$(12) \qquad u_{t+n+k} = \lambda u_{t+k} \quad (k = 0, 1, \cdots k) \qquad \text{where } \lambda = \frac{u_{t+n}}{u_t}, \text{ and}$$

$$v_{t+n+k} = \lambda v_{t+k} \quad (k = 1, 2, \cdots k).$$

By referring to theorem I, we have

$$A_{t+n}w_{t+n+1} + A_{t+n-1}v_{t+n+1} + A_{t+n-2}u_{t+n+1} = 1,$$

and consequently

$$\lambda(A_{t+n}w_{t+1} + A_{t+n-1}v_{t+1} + A_{t+n-2}u_{t+1}) = 1.$$

Therefore λ is a unit in the domain $k(\theta)$, i.e., $N(\lambda) = 1$. Furthermore, beginning with $k = 0$, $N(u_{t+k})$ forms a periodic series of numbers. Also, if we consider the rationalized complete quotient sets, we have that $(\alpha_k,\ \alpha'_k + \beta'_k\theta + \gamma'_k\theta^2,\ \alpha''_k + \beta''_k\theta + \gamma''_k\theta^2)$ is identical with $(\alpha_{k+n},\ \alpha'_{k+n} + \beta'_{k+n}\theta + \gamma'_{k+n}\theta^2,\ \alpha''_{k+n} + \beta''_{k+n}\theta + \gamma''_{k+n}\theta^2)$ for all values of $k > t$. To complete the proof we shall show that

$$(13) \qquad m_{t+k-1} = \frac{\beta'^3_{t+k} - D\gamma'^3_{t+k}}{(\beta'_{t+k}\gamma''_{t+k} - \beta''_{t+k}\gamma'_{t+k})^2}\alpha_{t+k},$$

and since the values of α_{t+k} and its constant multiplier are unchanged by increasing the subscript by n, it results that the series m_k is also periodic if $k \geqq t$.

By theorem I we have

$$\theta = \frac{B_k(\alpha''_{k+1}+\beta''_{k+1}\theta+\gamma''_{k+1}\theta^2)+B_{k-1}(\alpha'_{k+1}+\beta'_{k+1}\theta+\gamma'_{k+1}\theta^2)+B_{k-2}\alpha_{k+1}}{A_k(\alpha''_{k+1}+\beta''_{k+1}\theta+\gamma''_{k+1}\theta^2)+A_{k-1}(\alpha'_{k+1}+\beta'_{k+1}\theta+\gamma'_{k+1}\theta^2)+A_{k-2}\alpha_{k+1}},$$

$$\theta^2 = \frac{C_k(\alpha''_{k+1}+\beta''_{k+1}\theta+\gamma''_{k+1}\theta^2)+C_{k-1}(\alpha'_{k+1}+\beta'_{k+1}\theta+\gamma'_{k+1}\theta^2)+C_{k-2}\alpha_{k+1}}{A_k(\alpha''_{k+1}+\beta''_{k+1}\theta+\gamma''_{k+1}\theta^2)+A_{k-1}(\alpha'_{k+1}+\beta'_{k+1}\theta+\gamma'_{k+1}\theta^2)+A_{k-2}\alpha_{k+1}}.$$

Multiplying up and equating coefficients of like irrationalities, we get, dropping the subscripts on α, β, etc.,

$$
(14)\quad \begin{aligned}
C_k\alpha'' + C_{k-1}\alpha' + C_{k-2}\alpha &= DA_k\beta'' + DA_{k-1}\beta' = DB_k\gamma'' + DB_{k-1}\gamma',\\
B_k\alpha'' + B_{k-1}\alpha' + B_{k-2}\alpha &= C_k\beta'' + C_{k-1}\beta' = DA_k\gamma'' + DA_{k-1}\gamma',\\
A_k\alpha'' + A_{k-1}\alpha' + A_{k-2}\alpha &= B_k\beta'' + B_{k-1}\beta' = C_k\gamma'' + C_{k-1}\gamma'.
\end{aligned}
$$

If we eliminate α'' and α' from the first set of the above equations, we get

$$
(15)\quad \begin{vmatrix} C_k & C_{k-1} & DA_k\beta'' + DA_{k-1}\beta' - C_{k-2}\alpha \\ B_k & B_{k-1} & C_k\beta'' + C_{k-1}\beta' - B_{k-2}\alpha \\ A_k & A_{k-1} & B_k\beta'' + B_{k-1}\beta' - A_{k-2}\alpha \end{vmatrix} = 0 = \beta''\Delta_1 + \beta'\Delta_2 - \alpha,
$$

where

$$
\Delta_1 = \begin{vmatrix} C_k & C_{k-1} & DA_k \\ B_k & B_{k-1} & C_k \\ A_k & A_{k-1} & B_k \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} C_k & C_{k-1} & DA_{k-1} \\ B_k & B_{k-1} & C_{k-1} \\ A_k & A_{k-1} & B_{k-1} \end{vmatrix},
$$

and

$$
\begin{vmatrix} C_k & C_{k-1} & C_{k-2} \\ B_k & B_{k-1} & B_{k-2} \\ A_k & A_{k-1} & A_{k-2} \end{vmatrix} = 1.
$$

By use of the values of C_{k-1}, B_{k-1}, A_{k-1} obtained from the second group of equations (14), we can reduce Δ_1 and Δ_2 to functions of m_k and the coefficients α, β', $\cdots$ etc.

$$
\begin{aligned}
\Delta_1 &= -\frac{1}{\beta'}\begin{vmatrix} C_k & DA_k & DA_k\gamma'' + DA_{k-1}\gamma' - \beta''C_k \\ B_k & C_k & C_k\gamma'' + C_{k-1}\gamma' - \beta''B_k \\ A_k & B_k & B_k\gamma'' + B_{k-1}\gamma' - \beta''A_k \end{vmatrix}\\
&= -\frac{\gamma'}{\beta'}\begin{vmatrix} C_k & DA_k & DA_{k-1} \\ B_k & C_k & C_{k-1} \\ A_k & B_k & B_{k-1} \end{vmatrix}\\
&= -\frac{\gamma'}{\beta'^2}\begin{vmatrix} C_k & DA_k & DB_k\gamma'' + DB_{k-1}\gamma' - DA_k\beta'' \\ B_k & C_k & DA_k\gamma'' + DA_{k-1}\gamma' - C_k\beta'' \\ A_k & B_k & C_k\gamma'' + C_{k-1}\gamma' - B_k\beta'' \end{vmatrix}\\
&= -\frac{\gamma'\gamma''}{\beta'^2}m_k + \frac{\gamma''^2}{\beta''^2}\begin{vmatrix} C_k & DA_k & DB_{k-1} \\ B_k & C_k & DA_{k-1} \\ A_k & B_k & C_{k-1} \end{vmatrix}, \quad \left[\text{since } m_k = \begin{vmatrix} C_k & DA_k & DB_k \\ B_k & C_k & DA_k \\ A_k & B_k & C_k \end{vmatrix}\right]\\
&= -\frac{\gamma'\gamma''}{\beta'^2}m_k - \frac{\gamma'^2}{\beta'^3}\begin{vmatrix} C_k & DA_k & DC_k\gamma'' + DC_{k-1}\gamma' - DB_k\beta'' \\ B_k & C_k & DB_k\gamma'' + DB_{k-1}\gamma' - DA_k\beta'' \\ A_k & B_k & DA_k\gamma'' + DA_{k-1}\gamma' - C_k\beta'' \end{vmatrix}\\
&= -\frac{\gamma'\gamma''}{\beta'^2}m_k + \frac{\gamma'^3}{\beta'^3}D\Delta_1 + \frac{\gamma'^2\beta''}{\beta'^3}m_k.
\end{aligned}
$$

Solving for Δ_1,

$$
(13a)\qquad \Delta_1 = \frac{\gamma'^2\beta'' - \gamma'\gamma''\beta'}{\beta'^3 - D\gamma'^3}m_k.
$$

By a similar reduction

$$\Delta_2 = \frac{1}{\beta'}\begin{vmatrix} C_k & C_{k-1} & DB_k\gamma'' + DB_{k-1}\gamma' - DA_k\beta'' \\ B_k & B_{k-1} & DA_k\gamma'' + DA_{k-1}\gamma' - C_k\beta'' \\ A_k & A_{k-1} & C_k\gamma'' + C_{k-1}\gamma' - B_k\beta'' \end{vmatrix}$$

$$= \frac{\gamma''}{\beta'}\begin{vmatrix} C_k & C_{k-1} & DB_k \\ B_k & B_{k-1} & DA_k \\ A_k & A_{k-1} & C_k \end{vmatrix} + \frac{\gamma'}{\beta'}\begin{vmatrix} C_k & C_{k-1} & DB_{k-1} \\ B_k & B_{k-1} & DA_{k-1} \\ A_k & A_{k-1} & C_{k-1} \end{vmatrix} - \frac{\beta''}{\beta'}\Delta_1.$$

$$\Delta_2 = \frac{\gamma''}{\beta'}\Delta_3 + \frac{\gamma'}{\beta'^2}\begin{vmatrix} C_k & C_{k-1} & DC_k\gamma'' + DC_{k-1}\gamma' - DB_k\beta'' \\ B_k & B_{k-1} & DB_k\gamma'' + DB_{k-1}\gamma' - DA_k\beta'' \\ A_k & A_{k-1} & DA_k\gamma'' + DA_{k-1}\gamma' - C_k\beta'' \end{vmatrix}$$

$$= \left(\frac{\gamma''}{\beta'} - \frac{\beta''\gamma'}{\beta'^2}\right)\Delta_3 - \frac{\beta''}{\beta'}\Delta_1,$$

where

$$\Delta_3 = \begin{vmatrix} C_k & C_{k-1} & DB_k \\ B_k & B_{k-1} & DA_k \\ A_k & A_{k-1} & C_k \end{vmatrix}.$$

And reducing Δ_3,

$$\Delta_3 = \frac{1}{\beta'}\begin{vmatrix} C_k & DA_k\gamma'' + DA_{k-1}\gamma' - \beta''C_k & DB_k \\ B_k & C_k\gamma'' + C_{k-1}\gamma' - \beta''B_k & DA_k \\ A_k & B_k\gamma'' + B_{k-1}\gamma' - \beta''A_k & C_k \end{vmatrix}$$

$$= \frac{\gamma''}{\beta'}m_k + \frac{\gamma'}{\beta'}\begin{vmatrix} C_k & DA_{k-1} & DB_k \\ B_k & C_{k-1} & DA_k \\ A_k & B_{k-1} & C_k \end{vmatrix}$$

$$= \frac{\gamma''}{\beta'}m_k + \frac{\gamma'}{\beta'^2}\begin{vmatrix} C_k & DB_k\gamma'' + DB_{k-1}\gamma' - DA_k\beta'' & DB_k \\ B_k & DA_k\gamma'' + DA_{k-1}\gamma' - C_k\beta'' & DA_k \\ A_k & C_k\gamma'' + C_{k-1}\gamma' - B_k\beta'' & C_k \end{vmatrix}$$

$$= \left(\frac{\gamma''}{\beta'} - \frac{\gamma'\beta''}{\beta'^2}\right)m_k + \frac{\gamma'^2}{\beta'^2}\begin{vmatrix} C_k & DB_{k-1} & DB_k \\ B_k & DA_{k-1} & DA_k \\ A_k & C_{k-1} & C_k \end{vmatrix}$$

$$= \left(\frac{\gamma''}{\beta'} - \frac{\gamma'\beta''}{\beta'^2}\right)m_k + \frac{\gamma'^2}{\beta'^3}\begin{vmatrix} C_k & DC_k\gamma'' + DC_{k-1}\gamma' - DB_k\beta'' & DB_k \\ B_k & DB_k\gamma'' + DB_{k-1}\gamma' - DA_k\beta'' & DA_k \\ A_k & DA_k\gamma'' + DA_{k-1}\gamma' - C_k\beta'' & C_k \end{vmatrix}$$

$$= \left(\frac{\gamma''}{\beta'} - \frac{\gamma'\beta''}{\beta'^2}\right)m_k + \frac{\gamma'^3}{\beta'^3}D\Delta_3.$$

Hence

$$(13b) \qquad \Delta_3 = \frac{\beta'(\beta'\gamma'' - \beta''\gamma')}{\beta'^3 - D\gamma'^3}\,m_k,$$

and using the value of Δ_1, we have

$$\Delta_2 = \frac{\beta'\gamma'' - \beta''\gamma'}{\beta'^2}\Delta_3 - \frac{\beta''}{\beta'}\Delta_1$$

$$= \frac{\beta'\gamma'' - \beta''\gamma'}{\beta'^3 - D\gamma'^3}\,m_k\left[\frac{\beta'(\beta'\gamma'' - \beta''\gamma')}{\beta'^2} + \frac{\beta''\gamma'}{\beta'}\right],$$

and finally

$$(13c) \qquad \Delta_2 = \frac{\gamma''(\beta'\gamma'' - \beta''\gamma')}{\beta'^3 - D\gamma'^3} m_k.$$

Equation (15) becomes

$$(15') \quad -\beta''\gamma' \frac{(\beta'\gamma'' - \beta''\gamma')}{\beta'^3 - D\gamma'^3} m_k + \beta'\gamma'' \frac{(\beta'\gamma'' - \beta''\gamma')}{\beta'^3 - D\gamma'^3} m_k - \alpha = 0.$$

Solving for m_k and returning the dropped subscripts, we have

$$(13') \qquad m_k = \frac{\beta'^3_{k+1} - D\gamma'^3_{k+1}}{(\beta'_{k+1}\gamma''_{k+1} - \beta''_{k+1}\gamma'_{k+1})^2} \alpha_{k+1},$$

completing the proof of the theorem, as we obtain (13) by a change of subscripts.

The results of this last theorem afford an interesting geometrical proof that B_κ/A_κ and C_κ/A_κ actually converge towards θ and θ^2 respectively.

Consider the surface

$$x^3 + Dy^3 + D^2z^3 - 3Dxyz = m_k.$$

The straight line $x = \theta y = \theta^2 z$ is an asymptote to the surface, as can be seen by solving for their intersection, the line cutting the surface in three coincident points at infinity; that is, as x increases, the distance from any point on the surface to the line approaches zero, provided that m_κ is always finite, which condition is satisfied because of the last theorem. If $(C_\kappa, B_\kappa, A_\kappa)$ is any point on the surface, we have by the well-known formula for the distance from a point to a line

$$d^2 = C_\kappa^2 + B_\kappa^2 + A_\kappa^2 - \frac{(\theta^2 C_\kappa + \theta B_\kappa + A_\kappa)^2}{1 + \theta^2 + \theta^4},$$

which can be written

$$(1 + \theta^2 + \theta^4)d^2 = (C_\kappa - A_\kappa\theta^2)^2 + (B_\kappa - A_\kappa\theta)^2 + (C_\kappa\theta - B_\kappa\theta^2)^2.$$

We obtain

THEOREM VI. *If* 1, θ, θ^2 *expand into a ternary continued fraction, which ultimately becomes periodic, then*

$$\lim_{k=\infty} \frac{B_k}{A_k} = \theta \qquad \textit{and} \qquad \lim_{k=\infty} \frac{C_k}{A_k} = \theta^2.$$

Returning now to theorem V, we can show that "t" can not be 0 or 1.

THEOREM VII. 1, θ, θ^2 *can not be expanded into a purely periodic regular ternary continued fraction, nor into one with only one non-recurring partial quotient set.*

Suppose the expansion is purely periodic. Then by theorem I, since $\sigma_{1,\, n+1} = \theta$ and $\sigma_{2,\, n+1} = \theta^2$,

$$\theta = \frac{B_n\theta^2 + B_{n-1}\theta + B_{n-2}}{A_n\theta^2 + A_{n-1}\theta + A_{n-2}}.$$

Multiplying up and equating coefficients of θ^2, we have $A_{n-1} = B_n$; but $A_n > A_{n-1}$ which would make $A_n > B_n$. This is impossible according to our definition of a regular expansion.

Next suppose we have one non-recurring partial quotient set. Then

$$u_2 = -p_1 + \theta; \qquad v_2 = -q_1 + \theta^2; \qquad w_2 = 1;$$

and consequently

$$\sigma_{1,\, n+2} = \frac{\theta^2 - q_1}{\theta - p_1}; \qquad \sigma_{2,\, n+2} = \frac{1}{\theta - p_1};$$

and theorem I gives

$$\theta = \frac{B_{n+1}\left(\dfrac{1}{\theta - p_1}\right) + B_n\left(\dfrac{\theta^2 - q_1}{\theta - p_1}\right) + B_{n-1}}{A_{n+1}\left(\dfrac{1}{\theta - p_1}\right) + A_n\left(\dfrac{\theta^2 - q_1}{\theta - p_1}\right) + A_{n-1}}$$
$$= \frac{B_n\theta^2 + B_{n+1}\theta + (B_{n+1} - q_1B_n - p_1B_{n-1})}{A_n\theta^2 + A_{n+1}\theta + (A_{n+1} - q_1A_n - p_1A_{n-1})}.$$

Multiplying up and again equating coefficients of θ^2, we have $A_{n-1} = B_n$, which we have shown is impossible.

Experiment pointed towards expansions in which the number of non-recurring q's was not the same as the number of non-recurring p's. In that connection we have the following

THEOREM VIII. *If* 1, θ, θ^2 *expand into a periodic, regular ternary continued fraction with one non-recurring* q *and two non-recurring* p*'s and if the period contain* n *partial quotient sets, then* (C_n, B_n, A_n) *is a solution of the Pellian cubic* $x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1$.

Consider equations (12) for the special values $t = 2$, $k = 0$ and 1. They give $u_{n+2} = \lambda u_2$, $v_{n+3} = \lambda v_3$, where $N(\lambda) = 1$. From the third equation of (1)

$$\begin{aligned} w_{n+2} &= v_{n+3} + q_{n+2}u_{n+2}, \\ w_2 &= v_3 + q_2u_2. \end{aligned}$$

Since $w_2 = 1$, $u_{n+1} = w_{n+2}$, and we have assumed $q_{n+2} = q_2$,

$$u_{n+1} = \frac{w_{n+2}}{w_2} = \frac{v_{n+3} + q_2u_{n+2}}{v_3 + q_2u_2} = \lambda \tag{16}$$

or

$$N(u_{n+1}) = 1. \tag{16'}$$

The first conclusion we can draw is that the series of α's is purely periodic. It may be pointed out here that since from (4)

$$u_{n+1} = a_{n-2} + b_{n-2}\theta + c_{n-2}\theta^2,$$

$(a_{n-2}, b_{n-2}, c_{n-2})$ is a solution of the Pellian cubic for $m = 1$.

Having shown $N(u_{n+1}) = \alpha_{n+1} = 1$, we shall prove $m_n = 1$ by means of

$$(13d) \qquad m_n = \frac{\beta'^3_{n+1} - D\gamma'^3_{n+1}}{(\beta'_{n+1}\gamma''_{n+1} - \beta''_{n+1}\gamma'_{n+1})^2}\,\alpha_{n+1},$$

where $\alpha_{n+1} = 1$. Now

$$\begin{aligned} v_{n+1} &= u_{n+2} + p_{n+1}u_{n+1} \\ &= u_{n+1}(u_2 + p_{n+1}) \qquad \text{since } u_{n+2} = \lambda u_2, \text{ and } \lambda = u_{n+1}, \\ &= u_{n+1}(p_{n+1} - p_1 + \theta) \text{ since } u_2 \;= -p_1 + \theta, \end{aligned}$$

and it follows, since

$$\alpha'_{n+1} + \beta'_{n+1}\theta + \gamma'_{n+1}\theta^2 = \frac{v_{n+1}}{u_{n+1}} \times N(u_{n+1}),$$

that

$$(17) \qquad \alpha'_{n+1} = p_{n+1} = p_1; \qquad \beta'_{n+1} = 1; \qquad \gamma'_{n+1} = 0.$$

Similarly

$$\begin{aligned} w_{n+1} &= v_{n+2} + q_{n+1}u_{n+1} \\ &= u_{n+3} + p_{n+2}u_{n+2} + q_{n+1}u_{n+1} \\ &= u_{n+1}(u_3 + p_{n+2}u_2 + q_{n+1}), \end{aligned}$$

and since $u_2 = -p_1 + \theta$ and $u_3 = (p_1p_2 - q_1) - p_2\theta + \theta^2$, it follows by substitution as above that

$$(18) \quad \alpha''_{n+1} = (p_1p_2 - q_1 - p_1p_{n+2} + q_{n+1}); \quad \beta''_{n+1} = p_{n+2} - p_2; \quad \gamma''_{n+1} = 1.$$

Substituting the values found in (16), (17) and (18) in (13*d*), we have $m_n = 1$, and the theorem is proved.

Experimental attempts to find periodic expansions for the cube roots of integers led to the following definition of a NORMAL expansion. *If* $1, \theta, \theta^2$ *expand into a periodic regular ternary continued fraction with one non-recurring* q *and two non-recurring* p*'s, and if the period contain* n *partial quotient sets, and that part of the period indicated by* $p_2, q_2;\ p_3, q_3;\ \cdots\ p_n, q_n;$ *be skew-palindromic, and further* $p_{n+1} = p_{n+2} = p_1 + p_2$ *and* $q_{n+1} = p_1^2 + 2q_1$, *then the expansion is* NORMAL.

This definition was suggested by the following important theorem, which may be considered as the analogue of Lagrange's solution of $x^2 - Dy^2 = (-1)^{n-1}Q_n$, where Q_n is the nth denominator in the ordinary continued fraction expansion of $\sqrt{D}$.

THEOREM IX. *If* 1, θ, θ^2 *be expanded into a normal ternary continued fraction, then* $(C_{n-k+1}, B_{n-k+1}, A_{n-k+1})$ *is a solution of the Pellian cubic* $x^3+Dy^3+D^2z^3-3Dxyz=\alpha_k$ *for all values of* k.

We have assumed

$$p_k = p_{n-k+3}, \qquad q_k = q_{n-k+2} \tag{7}$$

and

$$p_{n+1} = p_{n+2} = p_1 + p_2, \qquad q_{n+1} = p_1^2 + 2q_1. \tag{19}$$

We shall show that under these conditions

$$u_k = u_{n+1}(C_{n-k+1} + B_{n-k+1}\theta + A_{n-k+1}\theta^2). \tag{20}$$

The first part of the proof is by induction. Assume equation (20) is true for all values up to and including k. From equation (6) by changing subscripts, we have

$$u_{k+1} = u_{k-2} - q_{k-1}u_{k-1} - p_k u_k, \tag{6''}$$

and successively substituting the values of u_{k-2}, u_{k-1}, and u_k assumed in (20) in (6''), we have

$$u_{k+1} = u_{n+1}\begin{bmatrix} C_{n-k+3} + & B_{n-k+3}\theta + & A_{n-k+3}\theta^2 \\ -\,q_{k-1}C_{n-k+2} - & q_{k-1}B_{n-k+2}\theta - & q_{k-1}A_{n-k+2}\theta^2 \\ -\,p_k\ C_{n-k+1} - & p_k\ B_{n-k+1}\theta - & p_k\ A_{n-k+1}\theta^2 \end{bmatrix}. \tag{21}$$

Consider the constant term in the bracket. If we apply equation (7) it becomes $C_{n-k+3} - q_{n-k+3}C_{n-k+2} - p_{n-k+3}C_{n-k+1} = C_{n-k}$ as can be verified by referring to the recursion formulæ (3). We have similar relations for the A's and B's, and equation (21) becomes

$$u_{k+1} = u_{n+1}(C_{n-k} + B_{n-k}\theta + A_{n-k}\theta^2). \tag{20'}$$

It follows that if equation (20) is true for k, it is also true for $k+1$, and hence if we show it is true for $k = 1, 2$ and 3, it will be true for all values of k.

In this connection we shall need the following set of equations, obtained from those of (14) by introducing the conditions found in (16), (17) and (18). We again omit for convenience the subscript $n+1$ on α, β', etc.

$$\begin{aligned} C_n\alpha'' + C_{n-1}\alpha' + C_{n-2} &= DB_n = DA_n\beta'' + DA_{n-1}, \\ B_n\alpha'' + B_{n-1}\alpha' + B_{n-2} &= DA_n = C_n\beta'' + C_{n-1}, \\ A_n\alpha'' + A_{n-1}\alpha' + A_{n-2} &= C_n = B_n\beta'' + B_{n-1}. \end{aligned} \tag{14'}$$

We have also $\alpha' = p_{n+1} - p_1$; $\alpha'' = p_1p_2 - q_1 - p_1p_{n+2} + q_{n+1}$; $\beta'' = p_{n+2} - p_2$, and the values of Δ_1, Δ_2 and Δ_3 as defined by (13*a*), (13*b*) and (13*c*) become $\Delta_1 = 0$, $\Delta_2 = 1$ and $\Delta_3 = 1$. We shall also make use of the following relations already deduced:

$$u_1 = 1, \qquad u_2 = -p_1 + \theta, \qquad u_3 = p_1p_2 - q_1 - p_2\theta + \theta^2$$

and

$$u_{n+1} = a_{n-2} + b_{n-2}\theta + c_{n-2}\theta^2.$$

By replacing u_{n+1} by its value, multiplying out and collecting coefficients, we get

$$\begin{aligned} u_{n+1}(C_n + B_n\theta + A_n\theta^2) &= P_1 + Q_1\theta + R_1\theta^2 \text{ say} \\ &= (C_n a_{n-2} + DB_n c_{n-2} + DA_n b_{n-2}) \\ &+ (C_n b_{n-2} + B_n a_{n-2} + DA_n c_{n-2})\theta \\ &+ (C_n c_{n-2} + B_n b_{n-2} + A_n a_{n-2})\theta^2. \end{aligned}$$

Writing the coefficients as determinants and reducing by means of (14′) as was done in theorem V, we have

$$R_1 = 0.$$

$$Q_1 = \begin{vmatrix} C_n & C_{n-1} & DA_n \\ B_n & B_{n-1} & C_n \\ A_n & A_{n-1} & B_n \end{vmatrix} = \Delta_1 = 0.$$

$$P_1 = \begin{vmatrix} C_n & C_{n-1} & DB_n \\ B_n & B_{n-1} & DA_n \\ A_n & A_{n-1} & C_n \end{vmatrix} = \Delta_3 = 1.$$

Hence equation (20) is true for $k = 1$, since we have just shown

$$1 = u_1 = u_{n+1}(C_n + B_n\theta + A_n\theta^2).$$

Similarly

$$\begin{aligned} u_{n+1}(C_{n-1} + B_{n-1}\theta + A_{n-1}\theta^2) &= P_2 + Q_2\theta + R_2\theta^2 \\ &= (C_{n-1}a_{n-2} + DB_{n-1}c_{n-2} + DA_{n-1}b_{n-2}) \\ &+ (C_{n-1}b_{n-2} + B_{n-1}a_{n-2} + DA_{n-1}c_{n-2})\theta \\ &+ (C_{n-1}c_{n-2} + B_{n-1}b_{n-2} + A_{n-1}a_{n-2})\theta^2. \end{aligned}$$

$$R_2 = 0.$$

$$Q_2 = \begin{vmatrix} C_n & C_{n-1} & DA_{n-1} \\ B_n & B_{n-1} & C_{n-1} \\ A_n & A_{n-1} & B_{n-1} \end{vmatrix} = \Delta_2 = 1.$$

$$P_2 = \begin{vmatrix} C_n & C_{n-1} & DB_{n-1} \\ B_n & B_{n-1} & DA_{n-1} \\ A_n & A_{n-1} & C_{n-1} \end{vmatrix} = \begin{vmatrix} C_n & C_{n-1} & DC_n - DB_n\beta'' \\ B_n & B_{n-1} & DB_n - DA_n\beta'' \\ A_n & A_{n-1} & DA_n - C_n\beta'' \end{vmatrix} = -\beta''\Delta_3 = -\beta''.$$

Hence $u_{n+1}(C_{n-1} + B_{n-1}\theta + A_{n-1}\theta^2) = -\beta'' + \theta$, while $u_2 = -p_1 + \theta$. Since $\beta'' = p_{n+2} - p_2$, it follows that in order that equation (20) be true $p_{n+2} = p_1 + p_2$, which we stated as a condition for a normal expansion.

And again $u_3 = (p_1p_2 - q_1) - p_2\theta + \theta^2$, while

$$\begin{aligned} u_{n+1}(C_{n-2} + B_{n-2}\theta + A_{n-2}\theta^2) &= P_3 + Q_3\theta + R_3\theta^2 \\ &= (C_{n-2}a_{n-2} + DB_{n-2}c_{n-2} + DA_{n-2}b_{n-2}) \\ &+ (C_{n-2}b_{n-2} + B_{n-2}a_{n-2} + DA_{n-2}c_{n-2})\theta \\ &+ (C_{n-2}c_{n-2} + B_{n-2}b_{n-2} + A_{n-2}a_{n-2})\theta^2. \end{aligned}$$

$R_3 = 1.$

$$Q_3 = \begin{vmatrix} C_n & C_{n-1} & DA_{n-2} \\ B_n & B_{n-1} & C_{n-2} \\ A_n & A_{n-1} & B_{n-2} \end{vmatrix} = \begin{vmatrix} C_n & C_{n-1} & -DA_n\alpha'' - DA_{n-1}\alpha' + DC_n \\ B_n & B_{n-1} & -C_n\alpha'' - C_{n-1}\alpha' + DB_n \\ A_n & A_{n-1} & -B_n\alpha'' - B_{n-1}\alpha' + DA_n \end{vmatrix}$$
$$= -\alpha''\Delta_1 - \alpha'\Delta_2 = -\alpha'.$$

Hence we must have $-\alpha' = -(p_{n+1} - p_1) = -p_2$ or $p_{n+1} = p_1 + p_2$, a condition of our hypothesis.

$$P_3 = \begin{vmatrix} C_n & C_{n-1} & DB_{n-2} \\ B_n & B_{n-1} & DA_{n-2} \\ A_n & A_{n-1} & C_{n-2} \end{vmatrix} = \begin{vmatrix} C_n & C_{n-1} & -DB_n\alpha'' - DB_{n-1}\alpha' + D^2A_n \\ B_n & B_{n-1} & -DA_n\alpha'' - DA_{n-1}\alpha' + DC_n \\ A_n & A_{n-1} & -C_n\alpha'' - C_{n-1}\alpha' + DB_n \end{vmatrix}$$
$$= -\alpha''\Delta_3 - \alpha'P_2 + D\Delta_1 = \alpha'\beta'' - \alpha''.$$

Hence we must have

$$\begin{aligned} p_1p_2 - q_1 &= \alpha'\beta'' - \alpha'' \\ &= (p_{n+1} - p_1)(p_{n+2} - p_2) - (p_1p_2 - q_1 - p_1q_{n+2} + q_{n+1}). \end{aligned}$$

Inserting the conditions $p_{n+1} - p_1 = p_2$ and $p_{n+2} - p_2 = p_1$, and solving for q_{n+1}, we have $q_{n+1} = p_1^2 + 2q_1$, a condition of the hypothesis. And hence it follows that equation (20) is true for $k = 3$, and hence for all values of k. Consequently, since $N(u_{n+1}) = 1$, the proposed theorem is true, that is $(C_{n-k+1}, B_{n-k+1}, A_{n-k+1})$ is a solution of $x^3 + Dy^3 + D^2z^3 - 3Dxyz = \alpha_k$.

It follows that the series of m's is purely periodic, its period being the reverse of the period of α_k. We may interpret n as the number of terms in any number of periods, for if the expansion is normal, then it is normal whether we consider only one period or any number of periods.

We state as a converse to theorem IX: *If the expansion of* $1, \theta, \theta^2$ *have one non-recurring* q *and two non-recurring* p*'s, after which it be periodic, and if* $\alpha_k = m_{n-k+1}$ *for all positive values of* k*, then the expansion is a normal expansion.*

We saw in theorem VIII that, under the conditions of the hypothesis, the series of α's is purely periodic, and that $N(u_{n+1}) = 1$. If $N(u_k) = N(C_{n-k+1} + B_{n-k+1}\theta + A_{n-k+1}\theta^2)$, then $u_k = \lambda(C_{n-k+1} + B_{n-k+1}\theta + A_{n-k+1}\theta^2)$, where $N(\lambda) = 1$. If we put $k = 1$, this reduces to $u_1 = 1 = \lambda(C_n + B_n\theta + A_n\theta^2)$, from which $\lambda = u_{n+1}$ and

$$u_k = (a_{n-2} + b_{n-2}\theta + c_{n-2}\theta^2)(C_{n-k+1} + B_{n-k+1}\theta + A_{n-k+1}\theta^2).$$

But we have seen that this equation can not be true for $k = 2$ and $k = 3$ unless $p_{n+1} = p_{n+2} = p_1 + p_2$ and $q_{n+1} = p_1^2 + 2q_1$. Further, we may write $u_{k+1} = u_{n+1}(C_{n-k} + B_{n-k}\theta + A_{n-k}\theta^2)$ in the form of equation (21), from which it follows that $p_k = p_{n-k+3}$ and $q_{k-1} = q_{n-k+3}$, and the converse is established.

We give as an illustrative example the expansion for $D = 13$, showing the expansion when u_n has not been rationalized, and when it has.

					A	B	C	
					1	0	0	
					0	1	0	
α	u	v	w	$p,\ q$	0	0	1	m
1	1	θ	θ^2	2, 5	1	2	5	8
5	$-2+\theta$	$-5+\theta^2$	1	1, 2	2	5	11	18
12	$-3-\theta-\theta^2$	$5-2\theta$	$-2+\theta$	1, 1	3	7	17	12
18	$8-\theta-\theta^2$	$1+2\theta-\theta^2$	$-3-\theta-\theta^2$	1, 1	6	14	33	5
8	$-7+3\theta$	$-11+2\theta^2$	$8-\theta-\theta^2$	1, 2	17	40	94	1
1	$-4-3\theta+2\theta^2$	$22-7\theta-\theta^2$	$-7+3\theta$	3, 14	259	609	1432	8
5	$34+2\theta-7\theta^2$	$49+45\theta-28\theta^2$	$-4-3\theta+2\theta^2$	3, 2	575	1352	3879	18
12	. . .	. . .	. . .	1, 1		. . .		12

$\alpha=\bar{u}$	$\bar{v}$	$\bar{w}$		
1	θ	θ^2	2, 5	as above
5	$6+3\theta-\theta^2$	$4+2\theta+\theta^2$	1, 2	
12*	$6+6\theta$	$8+2\theta+2\theta^2$	1, 1	
18*	$12+6\theta$	$3+3\theta+3\theta^2$	1, 1	
8	$7+3\theta-\theta^2$	$2+2\theta+2\theta^2$	1, 2	
1	$1+\theta$	$5+2\theta+\theta^2$	3, 14	
5	$16+3\theta-\theta^2$	$4+2\theta+\theta^2$	3, 2	
12	$6+6\theta$	$8+2\theta+2\theta^2$	1, 1	The period closes with the preceding set.

An attempt has been made to calculate a table of normal expansions. A table of expansions from 1 to 28 appears at the end of this paper. Because of the arbitrary choice of q, several expansions, all normal, have been found in many cases. They have not, however, all been recorded in this table.

The method of finding an expansion is largely experimental, and in most cases it is best to find a solution of the Pellian cubic for $m = 1$ first, and then to expand this solution. The rejection of incorrect choices of q_n is facilitated by theorem III, and the expansion is completed by the use of theorem IX. In the cases where a solution of the Pellian cubic is not available, these theorems help in finding such a solution.

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* Note here that $\bar{u}$, $\bar{v}$, $\bar{w}$ have a common factor, which is not removed, otherwise theorem IX would no longer hold.

TABLE OF NORMAL EXPANSIONS OF $\sqrt[3]{D}$ (EXCEPT AS INDICATED).

D	
2	1, 1; 2, 3; 3*, 3*;
3	1, 2; 0, 2; 1*, 5; 1, 2*;
4	1, 2; 0, 1; 0*, 1; 0, 1; 1, 5; 1, 1*; or 1, 1; 2, 1; 1*, 1; 3, 3; 3, 1*;
5	1, 2; 1, 1; 1*, 2; 3, 2; 1, 1; 2, 5; 2, 1*;
6	1, 2; 1, 1; 0*, 1; 1, 2; 0, 1; 0, 1; 0, 2; 1, 1; 0, 1; 2, 5; 2, 1*;
7[1]	1, 3; 0, 1; 0*, 1; 2, 6; 1, 1*;
9	2, 4; 4, 12; 6*, 12*;
10	2, 4; 4, 6; 3*, 6; 6, 12; 6, 6*;
11	2, 4; 4, 4; 2*, 4; 6, 12; 6, 4*;
12[1]	2, 5; 0, 2; 1*, 1; 0, 1; 1, 3; 2, 2; 2, 1; 1, 1; 2, 2; 1, 3; 6, 13; 2, 2*;
13	2, 5; 1, 2; 1*, 1; 1, 1; 1, 2; 3, 14; 3, 2*;
14	2, 5; 1, 2; 0*, 1; 0, 2; 3, 14; 3, 2*;
15	2, 6; 0, 2; 0*, 5; 0, 1; 0, 1; 0, 3; 0, 1; 0, 1; 0, 5; 0, 2; 2, 16; 2, 2*;
16	2, 6; 0, 1; 1*, 1; 1, 2; 2, 3; 1, 4; 1, 3; 2, 2; 1, 1; 1, 1; 2, 16; 2, 1*;
17	2, 5; 2, 1; 0*, 1; 0, 1; 0, 2; 5, 2; 0, 1; 0, 1; 0, 1; 4, 14; 4, 1*;
18	2, 6; 1, 1; 1*, 1; 2, 1; 1, 1; 3, 16; 3, 1*;
19	2, 6; 1, 1; 0*, 1; 0, 1; 0, 1; 0, 1; 0, 1; 0, 1; 3, 16; 3, 1*;
20	2, 6; 1, 1; 0*, 1; 0, 2; 1, 4; 1, 2; 0, 1; 0, 1; 3, 16; 3, 1*;
21	2, 6; 2, 1; 2*, 7; 1, 1; 0, 1; 0, 1; 1, 7; 2, 1; 4, 16; 4, 1*;
22	2, 6; 2, 1; 0*, 2; 1, 1; 0, 1; 1, 1; 1, 1; 0, 1; 1, 2; 0, 1; 4, 16; 4, 1*;
23	2, 6; 2, 1; 0*, 1; 2, 2; 0, 1; 0, 1; 1, 3; 0, 1; 0, 1; 1, 3; 0, 5; 1, 1; 12, 135; 12, 1; 1, 5; 0, 3; 1, 1; 0, 1; 0, 3; 1, 1; 0. 1; 0, 2; 2, 1; 0, 1; 4, 16; 4, 1*;
24	2, 7; 1, 1; 0*, 1; 3, 3; 0, 1; 0, 1; 0, 3; 3, 1; 0, 1; 3, 18; 3, 1*;
25[1]	2, 7; 1, 1; 0*, 1; 3, 7; 1, 1; 0, 2; 3, 3; 1, 1; 0, 4; 1, 2; 0, 1; 0, 1; 4, 6; 1, 1; 0, 1; 1, 2; 3, 3; 2, 1; 1, 7; 0, 1; 0, 4; 3, 2; 0, 1; 1. 7; 3, 1; 0, 1; 3, 18; 3, 1*;
26	2, 7; 1, 1; 0*, 1; 3, 18; 3, 1*;
28	3, 9; 6, 27; 9*. 27*;
29	
30	3, 9; 6, 9; 3*, 9; 9, 27; 9, 9*.

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* The first and last partial quotient set of the period are indicated by an asterisk.

[1] The expansion is not normal.